Lec 4: 08/31/2009

The familiar Fourier transform

$$f_{(k)} = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} f_{(x)}\, dx$$

is now just a change of basis $|x\rangle \to |k\rangle$:

$$f_{(k)} = \langle k|f\rangle = \int \langle k|x\rangle \langle x|f\rangle\, dx = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} f_{(x)}\, dx$$

At the first glance, Operator K seems to be Hermitian:

$$\langle x'| K |x\rangle = -i\,\delta'(x'-x)$$

$$\langle x| K |x'\rangle^* = +i\,\delta(x-x') = -i\,\delta(x'-x) = \langle x'| K |x\rangle$$

However, hermiticity requires that:

$$\langle g|K|f\rangle = \langle f|K|g\rangle^*$$

for all $|f\rangle, |g\rangle$. We have:

$$\langle g|K|f\rangle = \int_a^b \langle g|x\rangle \langle x| K|f\rangle\, dx =$$

$$\int_a^b g^*_{(x)} \frac{-i\,df_{(x)}}{dx}\, dx = -i\int_a^b g^*_{(x)}\, df_{(x)}$$

$$\langle f|K|g\rangle^* = \left[\int_a^b \langle f|x\rangle\langle x|K|g\rangle\, dx\right]^* =$$

$$\left[\int_a^b f^*_{(x)} \; -i\,\frac{dg_{(x)}}{dx}\; dx\right]^* = i\int_a^b dg^*_{(x)}\; f_{(x)}$$

Integrating by part:

$$i\int_a^b f_{(x)}\,dg^*_{(x)} = i f_{(x)}\, g^*_{(x)}\Big|_a^b - i\int_a^b g^*_{(x)}\, df_{(x)}$$

Thus:

$$\langle g|K|f\rangle = \langle f|K|g\rangle^* + i\, f_{(x)}\, g^*_{(x)}\Big|_a^b$$

K is only Hermitian if the last term (which is a surface term) vanishes.

Therefore, as long as $f_{(x)}=0$ at $x=a,b$ (or $f\to 0$ as $x\to\pm\infty$, if $a\to\infty$, $b\to-\infty$), K is a Hermitian operator. The eigenValues of K will be real, and this is why we chose $\langle x|k\rangle = \frac{1}{\sqrt{2\pi}}e^{ikx}$ with $k\in\mathbb{R}$.

There exists a subtlity when $f_{(x)} = \Psi_k(x)$ & $g_{(x)} = \Psi_{k'}(x)$. In this case, when the full interval $-\infty \leq x \leq +\infty$ is considered, hermiticity of K requires that:

$$\Psi^*_{k'}(x)\, \Psi_k(x) \Big|_{-\infty}^{+\infty} = e^{-ik'x}\, e^{ikx} \Big|_{-\infty}^{+\infty} = 0$$

This holds if $k = k'$. For $k \neq k'$, however, $e^{i(k-k')x}$ is not well defined as $x \to \pm\infty$ since it is an oscillating function. We can define:

$$\lim_{x \to \infty} e^{i(k-k')x} = \lim_{\substack{L \to \infty \\ \Delta \to \infty}} \frac{1}{\Delta} \int_{L}^{L+\Delta} e^{i(k-k')x}\, dx \longrightarrow 0$$

Physical Hilbert Space:

In quantum mechanics, we are interested in functions that are normalizable, i.e. can be normalized to unity. This requires that

$$|f(x)| = \int_a^b f^*(x)\, f(x)\, dx$$

be finite. However, eigenvectors of X and K are peculiar in that they are normalized to Dirac delta function:

$$\langle x' | x \rangle = \delta(x' - x) \qquad \langle k' | k \rangle = \delta(k' - k)$$

This happens when eigenvalues are continuous. We therefore consider the "physical Hilbert space", which consists of functions that can be normalized to "unity" or to the "Dirac delta function".